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Symmetry of the Parisi order-parameter space in spin glasses

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Abstract. We find and study a group of transformations of the Parisi order parameter. The spin-glass free energy is invariant under these transformations. A classification of infinitesimal transformations is given.

At the present time we have a good understanding of spin glasses at the mean-field level (see, for example, Binder and Young 1986, Mezard *et al* 1987). However, so far the problem of a phase transition in 3d spin glasses is open. Difficulties are related to an extremely complicated spectrum of order-parameter fluctuations in the low-temperature phase (De Dominicis and Kondor 1983, 1989; Goltsev 1984, 1986; Temesvári *et al* 1988). We think that a more profound understanding of the Parisi order-parameter space may be useful in this direction.

In the present paper we determine a group of transformation of the Parisi order parameter that do not change the spin-glass free energy.

At first we introduce the notion of the Parisi order-parameter space. Let us recall the principal properties of the Parisi order parameter (Parisi 1979, 1980).

The free energy of Ising spin glasses in the mean-field approximation has the following form (Sherrington and Kirkpatrick 1975, 1978):

$$F(q_{\alpha\beta})=\tfrac{1}{4}\beta\sum_{\alpha\beta}q_{\alpha\beta}^2-T\ln \operatorname{tr}_{\{S_\alpha\}}\exp\left(\tfrac{1}{2}\beta^2\sum_{\alpha\beta}q_{\alpha\beta}S_\alpha S_\beta+\beta H\sum_\alpha S_\alpha\right). \tag{1}$$

Here spins $S_\alpha=\pm1$, the order-parameter matrix $q_{\alpha\beta}$ is symmetric ($q_{\alpha\beta}=q_{\beta\alpha}$) with $q_{\alpha\alpha}=0$ at all α. Replica indices α and β run from 1 to n. Firstly, the order parameter $q_{\alpha\beta}$ has to satisfy the equation

$$\partial F/\partial q_{\alpha\beta}=0 \tag{2}$$

for each pair ($\alpha\beta$). Equation (2) is equivalent to the following equation: $q_{\alpha\beta}=\langle S_\alpha S_\beta\rangle$. Secondly, the free energy (1) should be stable with respect to small-order-parameter fluctuations (de Almeida and Thouless 1978). At temperatures T larger than a critical temperature T_c there is the only stable solution $q_{\alpha\beta}=q$, i.e. the replica-symmetric solution (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975, 1978; de Almeida and Thouless 1978). At $T<T_c$ the Parisi solution with broken replica symmetry is stable (Parisi 1979, 1980). A block structure of the Parisi order parameter $q_{\alpha\beta}$ is well known. Below we will call this matrix the canonical Parisi matrix ($q^c_{\alpha\beta}$). The value of a matrix element $q_{\alpha\beta}$ depends on an ultrametric distance $d_{\alpha\beta}\equiv\alpha\cap\beta$ between replicas α and β. If $d_{\alpha\beta}\neq d_{\gamma\nu}$ then $q_{\alpha\beta}=\tilde q\neq q_{\gamma\nu}=\tilde{\tilde q}$, that is $\langle S_\alpha S_\beta\rangle\neq\langle S_\gamma S_\nu\rangle$. From a physical point of view, correlation functions $\langle S_\alpha S_\beta\rangle$ and $\langle S_\gamma S_\nu\rangle$ should be equivalent. To restore

the symmetry between these two pairs of replicas $(\alpha\beta)$ and $(\gamma\nu)$ we can consider another matrix $q'_{\alpha\beta}$ which differs from the canonical matrix $q^c_{\alpha\beta}$ in a certain permutation of matrix elements (of course, matrix $q'_{\alpha\beta}$ should be a solution of equation (2)). In such a way for the considered replica pairs $(\alpha\beta)$ and $(\gamma\nu)$ we can obtain $q'_{\alpha\beta}=\langle S_\alpha S_\beta\rangle=\tilde{\tilde{q}}$ and $q'_{\gamma\nu}=\langle S_\gamma S_\nu\rangle=\tilde{q}$. Therefore, considering two matrices $q^c_{\alpha\beta}$ and $q_{\alpha\beta}$, we can say that the symmetry between $(\alpha\beta)$ and $(\gamma\nu)$ is restored. Then we can continue this procedure and restore the symmetry between all replica pairs $(\alpha\beta)$ (De Dominicis and Young 1983). Consequently we obtain a set of matrices $q'_{\alpha\beta}$ which differ from $q^c_{\alpha\beta}$ in all possible permutations of matrix elements. Below, this set of matrices is called the Parisi order-parameter space and is denoted by $P\{q_{\alpha\beta}\}$. The main aim of our paper is to study a symmetry of the space $P\{q_{\alpha\beta}\}$.

Let us find a set of transformations U which transform a matrix $q_{\alpha\beta}\in P\{q_{\alpha\beta}\}$ into another matrix $q'_{\alpha\beta}\in P\{q_{\alpha\beta}\}$:

$$q'_{\alpha\beta}=\sum_{\gamma\nu} U^{\alpha\beta,\gamma\nu}q_{\gamma\nu}. \tag{3}$$

It is clear that the free energy (1) has to be invariant under this transformation, i.e.

$$F(Uq)=F(q)=F(q^c). \tag{4}$$

Substituting (3) into (1) we find that equation (4) takes place if

$$U^TU=1 \tag{5}$$

$$\sum_{\gamma\nu} U^{\alpha\beta,\gamma\nu}S_\gamma S_\nu=S'_\alpha S'_\beta. \tag{6}$$

Equation (5) shows that $U^T=U^{-1}$, i.e. the transformation matrix U is a unimodular orthogonal matrix. In equation (6) $S'_\alpha=\pm1$ are new spin variables. We can satisfy equations (5) and (6) if the transformation U has the form

$$U^{\alpha\beta,\gamma\nu}=\tau^{\alpha\gamma}\tau^{\beta\nu} \tag{7}$$

where τ is a transformation τ: $(S_1,S_2,\ldots,S_n)\to(S'_1,S'_2,\ldots,S'_n)$, that is

$$S'_\alpha=\sum_\beta \tau^{\alpha\beta}S_\beta. \tag{8}$$

We find that the matrix $\tau^{\alpha\beta}$ may be presented in a form

$$\tau^{\alpha\beta}=\eta_\alpha\delta_{\pi(\alpha),\beta} \tag{9}$$

here $\eta_\alpha=\pm1$ for all α; $\delta_{\alpha,\beta}$ is the Kroneker symbol. A set of integer numbers $(\pi(1),\pi(2),\ldots,\pi(n))$ differs from the set $(1,2,\ldots,n)$ in a permutation of a few numbers. In other words, π is an operation of a permutation of the integer numbers $(1,2,\ldots,n)$, i.e. $\pi:\alpha\to\pi(\alpha)$. Below, the matrix (9) will be denoted $\tau(\pi)$. From (9) one obtains that $\tau^T\tau=1$ or $\tau^T=\tau^{-1}$. Consequently τ is a unimodular orthogonal matrix of the group SO(n). Substituting (7) and (9) into (3), we find

$$q'_{\alpha\beta}=\eta_\alpha\eta_\beta q_{\pi(\alpha)\pi(\beta)}. \tag{10}$$

Let us consider a case of a positive order parameter $(q_{\alpha\beta}>0)$. From a physical point of view this case takes place when a magnetic field is positive $(H>0)$. In this case we choose $\eta_\alpha=1$ at all α.

It is convenient to write the transformation (3) in matrix form

$$q'=\tau(\pi)q\tau^{-1}(\pi) \tag{11}$$

which resulted from (7) and the properties of the matrix $\tau(\pi)$ $(\tau^T=\tau^{-1})$.

It is interesting to note that the invariance of the spin-glass free energy (1) under the transformation (11) takes place at $H \neq 0$ because

$$\sum_{\alpha} S'_{\alpha} = \sum_{\alpha\beta} \tau^{\alpha\beta}(\pi) S_{\beta} = \sum_{\beta} S_{\beta}. \tag{12}$$

The set of the transformation $\{\tau(\pi)\}$ forms a subgroup of the group SO(n). This statement results from the relation $\tau(\pi_1)\tau(\pi_2) = \tau(\pi_1\pi_2)$.

To classify matrices τ we need now enter deeper into the Parisi hierarchical block procedure (Parisi 1979, 1980). At first we divide the sequence of numbers $1, 2, \ldots, n$ into blocks $I(j_1)$ where the block index $j_1 = 1, 2, \ldots, n/m_1$. Replica α belongs to block $I(j_1)$ if $(j_1 - 1)m_1 < \alpha \leqslant j_1 m_1$. Then each block $I(j_1)$ is divided into smaller blocks $I(j_1, j_2)$ where $j_2 = 1, 2, \ldots, m_1/m_2$ and m_2 is the number of replicas in each of these new blocks (replica α belongs to block $I(j_1, j_2)$ if we have $(j_1 - 1)m_1 + (j_2 - 1)m_2 < \alpha \leqslant (j_1 - 1)m_1 + j_2 m_2$. This procedure must be continued and each block $I(j_1, j_2)$ is divided into blocks $I(j_1, j_2, j_3)$ where $j_3 = 1, 2, \ldots, m_2/m_3$, and so on. The smallest blocks are denoted as $I(j_1, j_2, \ldots, j_R)$. Each replica α may be replaced by a sequence of hierarchical block numbers $\alpha = (j_1, j_2, \ldots, j_R, a)$ where $a = 1, 2, \ldots, m_R$ labels replicas in a smallest block $I(j_1, j_2, \ldots, j_R)$. Now we can determine the matrix element $q_{\alpha\beta}$. If $\alpha = (j_1, \ldots, j_R, a)$ and $\beta = (l_1, \ldots, l_R, b)$ where $j_1 = l_1, j_2 = l_2, \ldots, j_i = l_i$ but $j_{i+1} \neq l_{i+1}$ (i.e. $\alpha \in I(j_1 \ldots j_i j_{i+1})$ and $\beta \in I(j_1 \ldots j_i l_{i+1})$) then we have $q_{\alpha\beta} = q_i$. Using this hierarchy of blocks we can introduce a hierarchy of the transformations τ.

At first we introduce a set of transformations e_l which permute blocks $I(j_1 \ldots j_l j_{l+1})$ inside of the block $I(j_1 \ldots j_l)$, i.e. $e_l : (j_1 \ldots j_l\{j_{l+1}\}) \to (j_1 \ldots j_l\{\pi(j_{l+1})\})$. It is easy to show that $e_l q^{c} e_l^{-1} = q^{c}$. However, in general $e_l q e_l^{-1} \neq q$.

Now we introduce a notion of infinitesimal transformation u by saying that a transformation u is infinitesimal if

$$\| u q^{c} u^{-1} - q^{c} \|^2 \equiv \left| \sum_{\alpha\beta} ((u q^{c} u^{-1})_{\alpha\beta} - q^{c}_{\alpha\beta})^2 \right| \ll 1. \tag{13}$$

Let us consider a set of transformations $u(\pi_l) = u_l$, $l = 1, 2, \ldots, R$, where π_l permute only two blocks, for example, $I(j_1, \ldots, j_{l-1}, j_l, j_{l+1})$ and $I(j_1, j_2, \ldots, j_{l-1}, j'_l, j'_{l+1})$ where $j_l \neq j'_l$. In other words

$$\pi_l : \ldots I(j_1 \ldots j_l j_{l+1}) \ldots I(j_1 \ldots j'_l j'_{l+1}) \to \ldots I(j_1 \ldots j'_l j'_{l+1}) \ldots I(j_1 \ldots j_l j_{l+1}).$$

We find that

$$\| u_l q^{c} u_l^{-1} - q^{c} \|^2 = 8|m_l - m_{l+1}| m_l (q_{l-1} - q_l)^2|_{n \to 0} \to 0 \tag{14}$$

where at $l = R$, $m_{R+1} = 1$. In accordance with definition (13), the transformation u_l is infinitesimal. Therefore we obtain a hierarchy of infinitesimal transformations $\{u_1\}, \{u_2\}, \ldots, \{u_R\}$. It should be noted that a first attempt at the study of infinitesimal transformations has been made by Kondor and Nemeth 1987.

It is interesting to note that an arbitrary transformation τ may be presented as a product of infinitesimal operators u_l and e_l. For example, we consider a structure of a transformation $\tau(\pi(\alpha_0\beta_0)) \equiv \tau(\alpha_0\beta_0)$ where

$$\pi(\alpha_0\beta_0) : 1 \ldots \alpha_0 \ldots \beta_0 \ldots n \to 1 \ldots \beta_0 \ldots \alpha_0 \ldots n$$

that is $\pi(\alpha_0\beta_0)$ permutes two replicas α_0 and β_0 ($\alpha_0 \rightleftarrows \beta_0$). Let us have $\alpha_0 = (j_1, \ldots, j_R, a)$ and $\beta_0 = (l_1, \ldots, l_R, b)$ where $q_{\alpha_0\beta_0} = q_i$. Choosing a certain set of transformations u_l we can write the transformation $\tau(\alpha_0\beta_0)$ in the form

$$\tau(\alpha_0\beta_0) = u_{i+1}^{-1} u_{i+2}^{-1} \ldots u_{R-1}^{-1} u_R u_{R-1} \ldots u_{i+2} u_{i+1}. \tag{15}$$

In the limit $n \to 0$ we have

$$\|\tau(\alpha_0\beta_0)q^c\tau^{-1}(\alpha_0\beta_0)-q^c\|^2$$
$$=8\left|2q(x)\int_x^1 \mathrm{d}y\, q(y)-(1-x)q^2(x)-\int_0^1 \mathrm{d}y\, q^2(y)\right| \neq 0. \tag{16}$$

The last equation shows that in the Parisi order-parameter space the point $q' = \tau(\alpha_0\beta_0)q^c\tau^{-1}(\alpha_0\beta_0)$ is at a finite 'distance' from the point q^c.

Now we consider a sequence of transformations

$$\begin{aligned} &\tau_1 = u_{i+1}^{-1} \qquad \tau_2 = u_{i+1}^{-1}u_{i+2}^{-1} = \tau_1 u_{i+2}^{-1} \qquad \tau_3 = \tau_2 u_{i+3}^{-1} \qquad \ldots \\ &\tau_{R-i} = \tau_{R-i-1}u_R \qquad \tau_{R-i+1} = \tau_{R-i}u_{R-1} \qquad \ldots \\ &\tau_{2R-2i-1} = \tau_{2R-2i-2^u i+1} = \tau(\alpha_0\beta_0). \end{aligned} \tag{17}$$

This sequence of transformations τ_i generates a sequence of matrices $q_1, q_2, \ldots, q_{2R-2i-1}$ where $q_i \equiv \tau_i q^c \tau_i^{-1}$. Using equations (13), (15) and (16) we can find that, for $j \leqslant R-i$,

$$\|q_j - q_{j-1}\|^2 = \|\tau_j^{-1}\tau_{j-1}q^c\tau_{j-1}^{-1}\tau_j - q^c\|^2 = \|\mathrm{u}_{i+j}\mathrm{q}^c u_{i+j}^{-1} - q^c\|^2. \tag{18}$$

Therefore according to (14) the distance between points q_{j-1} and q_j tends to zero in the continuum limit $n \to 0$. The same result is obtained at $j > R-i$. It means that the sequence of matrices $q_1, \ldots, q_{2R-2i+1} = q'$ forms a continuous path from the point q^c to the point q' in the space $P\{q\}$.

The free-energy functional of a short-range spin glass has the form (Bray and Moore 1979)

$$\begin{aligned} F(q(\boldsymbol{r})) = \int \mathrm{d}\boldsymbol{r}[\tfrac{1}{4}\beta Spq^2(\boldsymbol{r}) - T\ln \mathrm{tr}_{\{S\}}\exp(\tfrac{1}{2}\beta^2 \boldsymbol{S}(\boldsymbol{r})q(\boldsymbol{r})\boldsymbol{S}(\boldsymbol{r})) \\ + \beta \boldsymbol{H}\boldsymbol{S}(\boldsymbol{r}) + (1/4z)\beta Sp(\nabla q(\boldsymbol{r}))^2] \end{aligned} \tag{19}$$

where z is the number of nearest neighbours. Moreover

$$\mathrm{Sp}\, q^2 = \sum_{\alpha\beta} q_{\alpha\beta}^2 \qquad \boldsymbol{S}q\boldsymbol{S} = \sum_{\alpha\beta} S_\alpha q_{\alpha\beta} S_\beta \tag{20}$$

where $\boldsymbol{S} = (S_1, S_2, \ldots, S_n)$. Free energy (19) is invariant under a global transformations $q'(\boldsymbol{r}) = \tau(\pi)q(\boldsymbol{r})\tau^{-1}(\pi)$. Moreover, two first terms in (19) are invariant under local transformations $q(\boldsymbol{r}) = \tau(\pi(\boldsymbol{r}))q(\boldsymbol{r})\tau^{-1}(\pi(\boldsymbol{r}))$.

Now there are interesting problems that still face us. The first problem is the problem of the topological structure of the space $P\{q\}$. The second problem is the problem of topological defects that can destroy the long-ranged order in short-range spin glasses.

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